Density Fluctuations Measurements in Supersonic Flows

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Turbulence in compressible flows plays a major role in a variety of aerospace applications. However, due to fluctuating thermodynamic quantities, compressible turbulence is challenging to analyze. Of these quantities, density fluctuations are of high importance due to their role in describing turbulence. A method to obtain quantitative measurements of density fluctuations has been developed using the optical technique of shadowgraph imaging. Shadowgraph imaging has frequently been used to obtain qualitative information about density gradients in fluids. However, quantitative application of these methods is much less common. The quantitative shadowgraph method presented here relates the light intensity variations in an image of the flow to the flow's density and applies statistical techniques to obtain quantitative information about it. Development of this technique allows measurements of compressible turbulence to be performed in the Missouri S&T supersonic wind tunnel.

I. Introduction

The study of turbulence is vitally important to many aerospace applications. From the drag on a small drone to the extreme environment of space re-entry, these processes are dominated by turbulence. In supersonic flows, turbulence becomes compressible, meaning the density of the flow (along with other thermodynamic quantities) fluctuates. As such, the analysis of compressible turbulence is significantly more difficult compared to incompressible (low speed) flows. Density fluctuations are particularly interesting because of the role they play in the production of turbulence.

Multiple methods have been developed by researchers to measure density fluctuations in supersonic flows. The most widely used is hot-wire anemometry. This intrusive (it needs to be physically placed into the flow) instrument can provide point-wise measurements of density fluctuations with high temporal resolution [1–3]. However, since hot-wire anemometers cannot directly measure density fluctuations, some assumptions on the flow are required in order to able to use this instrument.

Optical methods have the advantage in that they do not require the placement of a probe into the flow and are thus referred to as "non-intrusive". Various optical methods have been developed to measure density fluctuations in supersonic flows, each with its advantages and disadvantages. Most techniques rely on the change in the refractive index of a gas caused by changes in density, which affects how light is transmitted through the gas. Early methods used shadowgraph/Schlieren imaging [4–8], while more recent approaches rely on laser interferometry (such as Focused Laser Differential Interferometry [9]). Other methods rely on the change in a scattered light signal due to changes in number density (such as Filtered Rayleigh Scattering [10]). Among these techniques, shadowgraph imaging has the advantage of a relatively simple and inexpensive set-up, and a large field of view. Its primary disadvantage, however, is the fact that density measurements are averaged across the test section (and is thus two-dimensional).

Shadowgraph imaging is an optical method which uses light to visualize changes in refractive index and density gradients in transparent substances. Since its development in the 19th century, it has been frequently used for qualitative observation of density gradients in fluid flows. More recently, quantitative shadowgraph techniques have been developed to measure density fluctuations in fluid flows. However, quantitative analysis using the shadowgraph technique is much more demanding than qualitative analysis, due to the complex mathematical post-processing required to extract information from the shadowgraph images.

Development of a quantitative shadowgraph imaging method at S&T will allow the measurement and characterization of turbulent density fluctuations at the Missouri S&T supersonic wind tunnel and will greatly expand the measurement capabilities of the Aerodynamics Research Laboratory.

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II. Light Through a Medium

A. What is shadowgraph imaging?

Shadowgraph imaging is a method which visualizes optical inhomogeneities or disturbances in a transparent medium. For homogeneous gas, the refractive index is related to density through the Gladstone-Dale equation \[ \text{18} \]. Thus, in essence, shadowgraph imaging visualizes density gradients. This is done by passing a beam of light through the disturbance. The rays of light are bent according to the local refractive index. If gradients are present, the light will be bent differently, creating brighter and darker areas on the collected image, as shown qualitatively in figure 1. This method allows the easy visualization of density gradients in flows. An example shadowgraph image is shown in figure 2 where the thermal plume from a candle is shown. The hot combustion products from the flame have a lower density than the surrounding air, thus a strong density gradient is present at the interface between air and plume. Notice how the image is darker on the left side (negative density gradient), and brighter on the right side (positive density gradient).

![Fig. 1 Bending of light rays by optical inhomogeneity](image1)

![Fig. 2 A shadowgraph of the thermal plume from a candle](image2)

B. Mathematical relation between light intensity and index of refraction

Figure 2 provides a qualitative description of the flow, but if we want to extract quantitative data we first need to describe shadowgraph imaging mathematically.

Snell’s law, \[ \frac{n_1 \sin \theta_1}{n_2 \sin \theta_2} = \alpha \], describes how light bends upon moving from one medium to another. \( n_1 \) and \( n_2 \) are the refractive indices of the first and second mediums, respectively, and \( \theta_1 \) and \( \theta_2 \) are angles of incidence and exit. The amount that light is bent when transitioning from one material to another \( (\theta_2) \) depends on the ratio between the refractive indices of the two materials.

Snell’s law can be generalized to the case where the refractive index varies continuously. Settles describes the derivation of this relation, under the reasonable assumption that any light ray passing through the disturbance is only infinitesimally displaced inside the disturbance itself [11].

Thus, as stated in [11][12], for a refractive index gradient in the y-direction, refractive index and ray displacement...
are related by
\[
\frac{\partial^2 y}{\partial z^2} = \frac{1}{n} \frac{\partial n}{\partial y}, \quad \frac{\partial^2 y}{\partial z^2} = \frac{1}{n} \frac{\partial n}{\partial y} \tag{1}
\]

Then, for a ray that is bent through an angle \(\epsilon_y\) by the disturbance, the \(y\)-distance moved by the point where the ray hits the screen can be described by
\[
tan(\epsilon_y) = \frac{\Delta y}{l} \tag{2}
\]
where \(l\) is the horizontal distance from the disturbance to the screen. Therefore, as described by Merzkirch \[12\],
\[
\Delta y = l \tan(\epsilon_y) \tag{3}
\]

A similar equation can be developed for \(\Delta x\). By definition, \(\Delta x\) and \(\Delta y\) are equivalent to \(\frac{\partial x}{\partial z}\) and \(\frac{\partial y}{\partial z}\). By combining equations (1) with (3):
\[
\Delta y = l \tan(\epsilon_y) = \int \frac{1}{n} \frac{\partial n}{\partial y} \, dz \quad \Delta x = l \tan(\epsilon_x) = \int \frac{1}{n} \frac{\partial n}{\partial x} \, dz \tag{4}
\]

Light intensity in the absence of the inhomogeneity is described as a function of pixel location in the image, \(I = I(x, y)\). The light intensity distribution of the image when the inhomogeneity is present is then \(I^* = I^*(x^*, y^*)\). The process of tracing light rays from the initial distribution \(I\) to the region \(I^*\) produced by the ray bending can be considered as a coordinate transformation from \(I\) to \(I^*\). This mapping is correlated to the amount a ray was bent in order to produce the pattern of bright and dark regions on the screen \[12\]. Then, by the definition of a coordinate transformation,
\[
I^*(x^*, y^*) J = I(x, y) \tag{5}
\]
Where \(J\) is the Jacobian for the transformation:
\[
J = \left| \begin{array}{cc}
\frac{\partial x^*}{\partial x} & \frac{\partial x^*}{\partial y} \\
\frac{\partial y^*}{\partial x} & \frac{\partial y^*}{\partial y}
\end{array} \right| \tag{6}
\]

The result of the light ray deflections by the refractive index is the recorded shadowgraph image, \(I^*(x^*, y^*)\). The image \(I^*\) can be considered as the sum of all \(I_i(x, y)\), which are the original light intensities of each point \(x\) and \(y\) without the presence of the density disturbance. Following Merzkirch \[12\], by rearranging it is possible to obtain
\[
I^*(x^*, y^*) = \sum_i I_i(x, y) \tag{7}
\]

Assuming that all light rays bent when passing through the density gradient are only bent by an infinitesimal amount (a reasonable assumption), the new coordinates \((x^*, y^*)\) where a light ray hits the screen are only a small amount, \(\Delta\), away from the original coordinates \((x, y)\). Thus:
\[
x^* = x + \Delta x(x, y) \quad y^* = y + \Delta y(x, y) \tag{8}
\]
\(\Delta x\) and \(\Delta y\) are small enough to neglect products and powers.

Evaluating \(J\) using equations (8) gives
\[
J = 1 + \frac{\partial \Delta x}{\partial x} + \frac{\partial \Delta y}{\partial y} \tag{9}
\]

Combining equations (3) and (9)
\[
J = 1 + \frac{\partial}{\partial x} \left( l \int \frac{1}{n} \frac{\partial n}{\partial x} \, dz \right) + \frac{\partial}{\partial y} \left( l \int \frac{1}{n} \frac{\partial n}{\partial y} \, dz \right) \tag{10}
\]

Then, returning to equation (5) light intensity is seen to be:
\[
\frac{I(x, y)}{I^*(x^*, y^*)} = 1 + l \int \frac{1}{n} \frac{\partial^2 n}{\partial x^2} \, dz + l \int \frac{1}{n} \frac{\partial^2 n}{\partial y^2} \, dz \tag{11}
\]

This can be rearranged to give an equation relating the change in light intensity on the screen (contrast) to the second derivative of the refractive index.
\[
\frac{I - I^*}{I^*} = l \int \left( \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} \right) ln(n) \, dz \tag{12}
\]
C. Statistical properties of image contrast and refractive index

The processing of the recorded shadowgraph images to obtain quantitative density fluctuation data begins with determining the contrast (light intensity variation) of a shadowgraph image [11, 13]:

\[ h(x, y) = \frac{I(x, y) - I_{av}}{I_{av}} \] (13)

where \( h(x, y) \) is the contrast at any pixel \((x, y)\) in the image, \( I(x, y) \) is the light intensity (grey level) at that pixel, and \( I_{av} \) is the average grey level for all images. This contrast is the difference in illumination or light intensity of any given pixel in the image relative to the average background light intensity.

The relation between contrast and the second derivative of the refractive index [11] can be used to determine the relation between statistical properties of contrast and refractive index.

\[ h = L \int_{0}^{W} \left( \frac{\partial^{2} n}{\partial x^{2}} + \frac{\partial^{2} n}{\partial y^{2}} \right) dz \] (14)

To obtain information about density fluctuations, the quantities in the above equation can be broken into their mean and fluctuating components [7]. Considering only the fluctuating components,

\[ h' = L \int_{0}^{W} \left( \frac{\partial^{2} n'}{\partial x^{2}} + \frac{\partial^{2} n'}{\partial y^{2}} \right) dz \] (15)

To determine the statistical properties of contrast, the autocorrelation of both sides of Equation [14] must be taken. Autocorrelation is a statistical property which measures the similarity between a signal or data set and itself at a specific time or space lag. It can be used to analyze the similarity and randomness of the density fluctuations across the period of measurement.

The autocorrelation function for a continuous distribution in two dimensions is

\[ R_{xy}(\xi, \zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)f(x + \xi, y + \zeta) dx dy \] (16)

Equation [16] can be applied to both sides of Equation [14].

\[ R_{hh}(\xi, \zeta) = L^2 W^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial^{2} n'}{\partial x^{2}} + \frac{\partial^{2} n'}{\partial y^{2}} \right)^2 n(x, y, z)n(x + \xi, y + \zeta, z') dx dy \] (17)

Using various integral properties, this equation can be reconfigured in terms of \( R_{nn} \), the autocorrelation of refractive index [14]. Further simplification and manipulation yields a relation between the autocorrelation of contrast, \( R_{hh} \), and the autospectra of refractive index, \( S_{nn} \). Then, using the fact that the autospectra of a function is the Fourier transform of its autocorrelation, an equation relating autospectra of contrast, \( S_{hh} \), and autospectra of refractive index, \( S_{nn} \) can be obtained. The autospectra allows the analysis of possible periodicity of fluctuations within the flow. The equations relating \( R_{hh} \) and \( R_{nn} \), and \( S_{hh} \) and \( S_{nn} \) can then be solved for \( R_{nn} \) and \( S_{nn} \), the statistical properties of the refractive index. Variance can also be determined using the equation for \( R_{nn} \) and the relationship between autocorrelation and variance. The variance of density fluctuations can be used to determine the intensity and size of density fluctuations in the flow.

The Gladstone-Dale equation can be used to relate refractive index to density:

\[ n - 1 = K \rho \] (18)

where \( n \) is the index of refraction, \( \rho \) the local density, and \( K \) the Gladstone-Dale coefficient, which is a function of the specific gas, temperature, and wavelength of light used [11, 12, 15].

Thus it is possible to obtain statistical information about density fluctuations from the light intensity information found in the shadowgraph images.
III. Experimental Set-Up

A. The Missouri S&T Supersonic Wind Tunnel

The Missouri S&T Supersonic Wind Tunnel (shown in Fig. 3), located in the Aerodynamics Research Laboratory, is a blow-down, cold-flow facility originally commissioned in 1968 [16]. In its original configuration it was equipped with an axisymmetric, Mach 3 nozzle with an exit diameter of 127 mm. Nominal stagnation pressure and temperature are 1.52 MPa and 280 K respectively. These provide a Reynolds’ number per unit length of $1.35 \times 10^8 \ 1/m$, which is among the highest for similar tunnels used in universities. The stagnation pressure is maintained within $\pm 3\%$ of its nominal value by a LabView-based, scheduled-gain PID controller. The total temperature in the plenum varies by only $\pm 1\%$ during a 20-second test run. The long piping system between the main reservoir and the wind tunnel effectively acts as a heat exchanger. The main facility reservoir has a volume of 7.704 m$^3$ and can be pressurized up to 13.8 MPa, providing run times up to 1 minute (at nominal stagnation pressure). The original free-jet test section had a diameter and length of 387.4 mm and 593.8 mm respectively. Optical access was limited to two opposing 323.9 mm diameter windows perpendicular to the flow. These windows lie relatively far from the flow, thus providing very limited view angles. A model support system and lead plate for instrumentation were also available. Although the data presented in this work were collected in its original configuration, the facility is currently undergoing a major upgrade with a new nozzle, test section and diffuser. A new Tomographic Particle Image Velocimetry system is now also available [17].

![Fig. 3 Missouri S&T’s supersonic wind tunnel in its original configuration](image)

B. Shadowgraphy Equipment

A z-type mirror shadowgraphy system (so-called because of the system shape) was used to obtain the images. This is shown in figure 4. One 6-inch spherical mirror with focal length of 58 inches was placed on either side of the wind tunnel test section. The light source, using a Prizmatix white light source controller, was positioned immediately beside the tunnel. The light source housing was equipped with adjustable cut-offs to change the size and shape of the exiting light beam.
beam of light. This was adjusted to approximate a point light source. The first mirror was positioned exactly one focal length away from the light source in order to collimate (make parallel) the beam of passing through the test section. Collimation greatly simplifies the analysis of the shadowgraph images. The camera was positioned exactly one focal length away from the second mirror so that the light beam would be focused into the camera and no light would be lost. Offset angles between both mirrors and the line normal to the flow were minimized in order to reduce errors and aberrations resulting from the mirrors being used in an off-axis configuration [11]. An i-SPEED 727 high speed camera was used to record shadowgraph images at 10,000 frames per second.

![Fig. 4 Shadowgraphy setup](image)

IV. Results

A. Shadowgraph image data processing

Once the system was set up and all optical components were aligned, a calibration image (Figure 5) was taken of a caliper set at 5 mm in the test section. The same camera settings were used for the calibration image and the shadowgraph test, allowing for the determination of the size of each pixel in the image in millimeters. Five millimeters are equal to 39 pixels in the image, and so each pixel is approximately 0.256 mm. The knowledge of pixel size can be used to determine the scale of fluctuations in the shadowgraph images.

The wind tunnel was run and the camera remotely triggered to record three seconds of data in the middle of the test so that uniform flow conditions would be present throughout the images. One second of data was saved, yielding 10,000 images for analysis. These were saved as .jpg images for ease of processing. Figure 6 provides an example of one such shadowgraph image.

Initial post-processing of images included working in MATLAB to convert images to 8-bit grey scale (256 grey levels) and to crop them so that only the relevant part of the image was included. Average grey level was calculated over all the images. Then, pixel contrast for each image was calculated according to Equation 13 and contrast arrays were saved as .mat files.

Autocorrelation for contrast was calculated using the autocorrelation equation for discontinuous data:

\[
R(\xi, \zeta) = \frac{\langle H(x, y)H(x + \xi, y + \zeta) \rangle}{\langle H(x, y)^2 \rangle} \tag{19}
\]

where the brackets denote ensemble averages across all images. Results were plotted using MATLAB. The autocorrelation plots for a specific pixel in the x and y directions are shown in figure 8.
B. Images

The images obtained were of good quality and provide plentiful data for further analysis. Current analysis extends to image contrast and autocorrelation of image contrast data.

The average shadowgraph image shown in figure 7b shows distinct lines on a background with a fairly even grey level. The lines in this image and each individual shadowgraph image (such as figure 6) are Mach waves due to small imperfections in the supersonic nozzle. That the lines denoting Mach waves are still distinct in the average image is evidence that these Mach waves do not significantly change position with time throughout the image data.

It can be seen in Figure 7c that levels of image contrast are homogeneous throughout the shadowgraph image, that is, light intensity fluctuations are similar across the image.

Figure 8 shows the autocorrelation plot in two directions, horizontal and vertical. The autocorrelation plot shows that the density fluctuations are different in the two directions, and thus not isotropic. This information is critical, since one of the assumptions used in the work of Clay [5] and Hermann [6] to calculate density fluctuations was that of isotropic turbulence. It is clear that this is not valid and it should be accounted for in future developments.

V. Conclusions

The measurement of density fluctuations in compressible turbulence is of great import to the study of and understanding of turbulence in many research areas. To this end, a shadowgraph imaging technique is being developed for use at the Missouri S&T supersonic wind tunnel. This will enable the characterization of free-stream turbulence in the wind tunnel, and will also add a valuable tool to the measurement techniques available in the Aerodynamics Research Lab. To date, much literature review has been performed to determine system set-up and to develop a data processing technique for the shadowgraph images that were recorded. Initial data processing yields measurements of image contrast, and will be continued to determine the statistical properties of density which will be analyzed to better understand the turbulent flow.
Fig. 6  Sample shadowgraph image (flow from right to left)

Fig. 7  Shadowgraph images processing, a) example of instantaneous image b) average of all images c) example of contrast image.
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References


Reflection on Learning Experience

Allie Dingfield

After participating in the OURE program during the 23/24 school year, I gained much knowledge about the process of conducting research. This program gave valuable development to my skills as a researcher, skills which can now be applied other areas of my academic studies.

A very important initial step to conducting research is to perform a literature review. There are multitudes of references and existing research available. It is vital to investigate what previous work has been done regarding one’s research topic, what methods those researchers used, why they used those methods, what assumptions they made, and what their data looked like. This aids in the understanding of the research topic and allows the critical consideration of the problem so that the best way to investigate the problem can be determined. Then, using knowledge gained from the literature review, the experiment can be planned and performed. During this process, input from other researchers should be considered to prevent oversights or misconceptions. After the experiment is conducted, a clear path and procedure should be followed for data processing. This method should be outlined prior to conducting the experiment so that all necessary data is collected. The experimental results can then be analyzed, and conclusions can be made.

Throughout my research project, I became familiar with using Google Scholar, online library resources, and physical library resources to find papers and books as references relevant to my research. I learned the various works cited in a given paper or book are a valuable source to locate other papers which will yield more information on a given topic. Existing research was a very helpful resource for me as I developed my research project.

In designing an experiment, I learned it is important to have a good general understanding of the entire research process for the problem before beginning experimental design. One needs to start with the theoretical and mathematical foundation of the problem to understand why certain experimental parameters need to be set the way they are, and how and why changing one parameter will impact the experiment. It is important to think through components needed and necessary steps as the experiment is designed. It is also vital to be open-minded and not set on doing the experiment one specific way, else experimental options that may be much easier and or yield better data may not be considered.

To interpret experimental results, it is important to draw on the theoretical and mathematical foundation of the problem. Referencing existing research can help in this. Comparing the results with expected results and working to determine the reasons for any differences also yields valuable information.