Tempered Fractional Brownian Motion with Reflecting Walls

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Background

Anomalous diffusion:

- Random motion that violates linear relation between mean-square displacement and time, $\langle x^2 \rangle \sim t$
- Can be caused by long-time correlations between steps
- Can be modeled mathematically by Fractional Brownian Motion (FBM)

Reflected FBM

- Recent results [1]: unusual accumulation and depletion of particles close to reflecting walls (Normal diffusion=Flat)
- Probability density features power-law singularity $P(x) \sim x^{\kappa}$ at the wall

Tempered FBM:

- In many applications, correlations between steps have a large but finite range
- For tempered FBM, power-law correlations are cut off (tempered) beyond a certain "tempering" time scale [2]

Objective

- Study tempered FBM on a one-dimensional interval with reflecting walls
- Analyze how tempering affects accumulation and depletion of particles near the walls
- Determine functional form of probability density close to a reflecting wall

Fractional Brownian Motion

- Gaussian stochastic process with stationary, power-law correlated increments
- Discrete FBM: $x_{n+1} = x_n + \xi_n$ (Xi)
- Covariance:

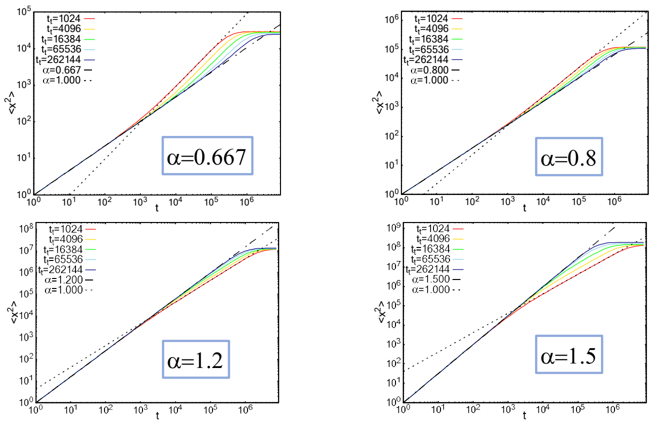
$\langle \xi_m \xi_{m+n} \rangle = C_n = \frac{1}{2}\sigma^2[|n-1|^{\alpha} - 2|n|^{\alpha} + |n+1|^{\alpha}]$ (behaves as power law at large n)

- α: anomalous diffusion exponent (0 < α < 2) (Normal Diffusion speed)
- Tempering $\langle \xi_m \xi_{m+n} \rangle = C_n \exp(-\frac{t}{t_t})$ (Kills Correlations)

Monte Carlo methods

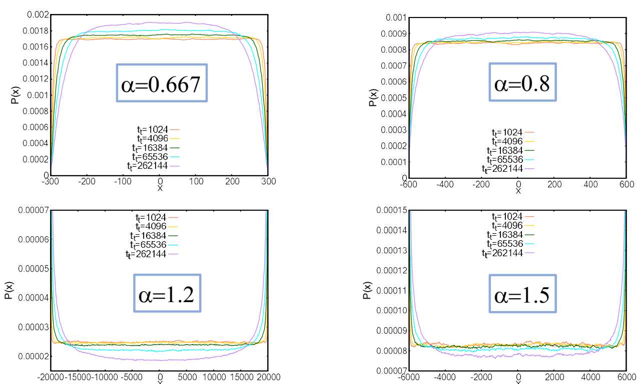
- Particles move on interval [-*L*,*L*] with reflecting walls at both ends
- Particles start at origin at time $t = 0$ (Define Wall)
- Correlated random numbers representing ξ created via Fourier filtering technique [3]

Results: Mean-square displacement



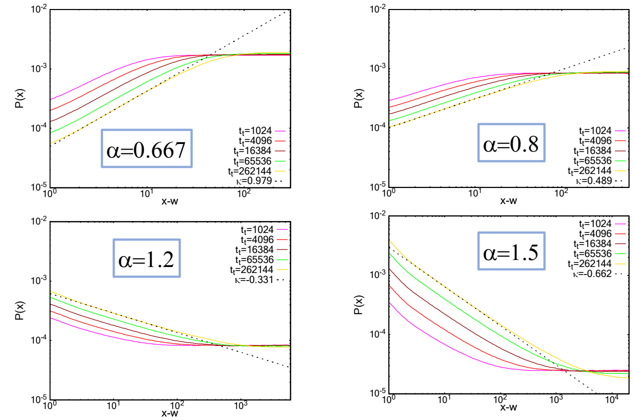
- Mean-square displacement vs. time for different α and tempering times t_t (Measures square of displacement)

Results: Probability density



- Stationary probability density for different α. (Normal Diffusive Case in center due to tempering)

Results: Behavior near wall



- Stationary probability density vs distance from the wall Dashed lines are power-law fits to $P(x) \sim (x-w)^{\kappa}$ (Log-Log, power law flat, all power laws same exponent)

Conclusions

- Studied impact of tempering time on FBM on one-dimensional interval with reflecting walls
- Particle accumulation and depletion near reflecting walls as for normal FBM
- Width of the accumulation/depletion region controlled by tempering time
- Power law singularity in probability density close to wall, $P(x) \sim (x-w)^{\kappa}$ with $\kappa = (2/\alpha) - 2$ (as in the untampered case)

References + Funding

[1] A.H.O. Wada and T. Vojta, Phys. Rev E 97, 020102 (2018)
T. Guggenberger, et al., New J. Phys. 21, 022002 (2019)
[2] D. Molina-Garcia et al, New J. Phys. 20, 103027 (2018)
[3] H. A. Makse et al., Phys. Rev. E 53, 5445 (1996)

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